A power-law upper bound on the correlations in the two-dimensional random-field Ising model

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Random-field Ising model

- Standard Ising model:
  Domain $\Lambda \subset \mathbb{Z}^d$. Boundary conditions τ outside Λ.
  Energy of configuration $\sigma: \Lambda \to \{-1,1\}$ given by

$$H^{\Lambda,\tau}(\sigma) = -J \sum_{\substack{u \sim v, \\ u,v \in \Lambda}} \sigma_u \sigma_v - J \sum_{\substack{u \sim v, \\ u \in \Lambda, v \notin \Lambda}} \sigma_u \tau_v - h \sum_{v \in \Lambda} \sigma_v$$

- At temperature T:

$$\mathrm{Prob}(\sigma) \propto \exp\left(-\frac{1}{T} H^{\Lambda,\tau}(\sigma)\right)$$

- Random-field Ising model (RFIM):

$$H^{\Lambda,\tau}(\sigma) = -J \sum_{\substack{u \sim v, \\ u,v \in \Lambda}} \sigma_u \sigma_v - J \sum_{\substack{u \sim v, \\ u \in \Lambda, v \notin \Lambda}} \sigma_u \tau_v - \varepsilon \sum_{v \in \Lambda} \eta_v \sigma_v$$

  with (η_v) a quenched random field.

- In this talk – (η_v) independent standard Gaussians.

Long-range order

- The Ising model, at $h = 0$, exhibits long-range order at low temperatures.
- Is this the case also for the random-field Ising model?
- No, when ε is large! (strong disorder regime)
- Proof for $T = 0$:

If $|\eta_v| > 2d \cdot \frac{J}{\varepsilon}$ then necessarily $\mathrm{sign}(\sigma_v) = \mathrm{sign}(\eta_v)$.

- At large ε, such vertices are likely to separate the origin from the boundary of $\Lambda(\mathrm{L})$. Thus

$$\mathbb{E}\left[< \sigma_0 >_T^{\Lambda(L),+}\right] \xrightarrow[L\to\infty]{} 0$$

with convergence occurring exponentially fast in L.

- Recent more quantitative results by Camia-Jiang-Newman(18).

Imry-Ma phenomenon

- Imry-Ma (75) considered small ε (weak disorder) and argued that:
  - Long-range order occurs in dimensions $d \geq 3$.
  - No long-range order in two dimensions:
  Unique Gibbs state for all $T \geq 0$. Even for arbitrarily weak disorder!

- An essence of the argument:
  With plus bounday conditions,
  is the plus configuration favored over the minus configuration?
  Energy difference is

$$H^{\Lambda(L),+}(+) - H^{\Lambda(L),+}(-) \approx \mathrm{J} \cdot L^{d-1} \pm \varepsilon \cdot L^{\frac{d}{2}}$$

  Boundary wins when $d \geq 3$.
  Random field wins, due to random fluctuations, when $d = 2$.

- Proofs. $\mathrm{d} \geq 3$: Imbrie ($T = 0$, 85), Bricmont-Kupiainen (88)

  $d = 2$: Aizenman-Wehr (89)

(quantum: Aizenman-Greenblatt-Lebowitz 09)

Rate of decay of boundary effect

- How fast does the boundary effect decay in two dimensions?
  How large is $\mathbb{E}\left[< \sigma_0 >_T^{\Lambda(L),+}\right]$?
- **Main result** (power-law upper bound):
  In two dimensions, for any $\mathrm{T} \geq 0,\ J, \varepsilon > 0$,

$$\mathbb{E}\left[< \sigma_0 >_T^{\Lambda(L),+}\right] \leq \frac{1}{L^\gamma} \quad \text{for large L}$$

  the obtained power γ is very small, behaving as

$$\gamma \approx \exp\left(-c\ \left(\frac{J}{\varepsilon}\right)^2\right) \quad \text{for small } \frac{\varepsilon}{J}$$

- **Corollary** (by FKG inequality):
  A similar power-law upper bound for correlations in the RFIM
- Improves Chatterjee (17) $\frac{1}{\sqrt{\log(\log(L))}}$ decay.

Ideas of proof for $T = 0$

- Denote ground-state configuration by $\sigma^{\Lambda,\tau}$.
- Influence-percolation: $P_L \coloneqq \mathbb{E}\left[\sigma_0^{\Lambda(L),+}\right] = \mathbb{P}\left(\sigma_0^{\Lambda(L),+} > \sigma_0^{\Lambda(L),-}\right)$
  Main result: Power-law upper bound on P_L.
- First observable: The number of sites in $\Lambda(\ell)$ influenced by boundary conditions on $\Lambda(3\ell)$

$$D_\ell(\eta) := \left|\left\{v \in \Lambda(\ell) : \sigma_v^{\Lambda(3\ell),+}(\eta) > \sigma_v^{\Lambda(3\ell),-}(\eta)\right\}\right|$$

- Note: Using FKG inequality, $\mathbb{E}[D_\ell(\eta)] \geq \ell^2 \cdot P_{4\ell}$.
- Second observable: Work in annulus $\Lambda(3\ell) \setminus \Lambda(\ell)$ with $+$ or $-$ boundary conditions inside and outside.
- Ground-state energies $\mathcal{E}^{+,+}, \mathcal{E}^{-,-}, \mathcal{E}^{+,-}, \mathcal{E}^{-,+}$. Functions of field η.
- Surface tension: $\tau_\ell(\eta) \coloneqq -(\mathcal{E}^{+,+} + \mathcal{E}^{-,-} - \mathcal{E}^{+,-} - \mathcal{E}^{-,+})$.

Main steps

- **Step 1** (upper bound): $\mathbb{E}[\tau_\ell(\eta)] \leq CJ \cdot \ell \cdot P_{\ell-1}$.
- **Step 2** (exact expression):

$$\mathbb{E}[\tau_\ell(\eta)] = \frac{2\varepsilon}{\ell} \int_{-\infty}^{\infty} \mathbb{E}[D_\ell(\eta^t)]\, dt$$

with $\eta^t \equiv \eta + \frac{\mathrm{t}}{\ell}$ inside $\Lambda(\ell)$,
$\eta^t \equiv \eta$ outside $\Lambda(\ell)$.

Note: the sum $\sum_{v\in\Lambda(\ell)} \eta_v$ increases by t standard deviations in η^t

- Put together, these imply the anti-concentration bound

$$\mathbb{P}\left(\frac{D_\ell}{\mathbb{E}(D_\ell)} < \frac{1}{2}\right) \geq \mathbb{P}\left(|N(0,1)| > C \cdot \frac{J}{\varepsilon} \cdot \frac{P_{\ell-1}}{P_{4\ell}}\right)$$

Variance bound

- Anti-concentration bound:

$$\mathbb{P}\left(\frac{D_\ell}{\mathbb{E}(D_\ell)} < \frac{1}{2}\right) \geq \mathbb{P}\left(|N(0,1)| > C \cdot \frac{J}{\varepsilon} \cdot \frac{P_{\ell-1}}{P_{4\ell}}\right)$$

- Right-hand side is constant when P_ℓ approximately a power of ℓ.
- **Step 3:** This is contrasted with a variance bound:
  If $P_\ell \approx \frac{1}{\ell^\delta}$ then $\mathrm{Var}(D_\ell) \leq C \cdot \delta \cdot \left(\mathbb{E}(D_\ell)\right)^2$.
- Chebyshev's inequality implies that $\mathbb{P}\left(\frac{D_\ell}{\mathbb{E}(D_\ell)} < \frac{1}{2}\right) < C \cdot \delta$
- Contradiction arises if δ is too small.

Step 1: surface tension upper bound

- Claim: $\mathbb{E}[\tau_\ell(\eta)] \le CJ \cdot \ell \cdot P_{\ell-1}$
  with $\tau_\ell(\eta) \coloneqq -(\mathcal{E}^{+,+} + \mathcal{E}^{-,-} - \mathcal{E}^{+,-} - \mathcal{E}^{-,+})$.

- Proof: Let $\sigma^{s,s'}$ be the ground state in $\Lambda(3\ell) \setminus \Lambda(\ell)$ subject to s, s' boundary conditions inside and outside. Then $\mathcal{E}^{s,s'}$ is its energy.

- Form mixed configurations $\tilde{\sigma}^{+,-}$ and $\tilde{\sigma}^{-,+}$:
  $$\tilde{\sigma}^{s,s'} \equiv \sigma^{s,s} \quad \text{on } \Lambda(3\ell) \setminus \Lambda(2\ell)$$
  $$\tilde{\sigma}^{s,s'} \equiv \sigma^{s',s'} \quad \text{on } \Lambda(2\ell) \setminus \Lambda(\ell)$$
  and write $\tilde{\mathcal{E}}^{s,s'}$ for their energy with s, s' boundary conditions.

- Of course, $\mathcal{E}^{+,-} \le \tilde{\mathcal{E}}^{+,-}$ and $\mathcal{E}^{+,-} \le \tilde{\mathcal{E}}^{+,-}$ by def. of ground state.
  Thus $$\tau_\ell(\eta) \le -\left(\mathcal{E}^{+,+} + \mathcal{E}^{-,-} - \tilde{\mathcal{E}}^{+,-} - \tilde{\mathcal{E}}^{+,-}\right)$$

- The sole contribution to the right-hand side comes from the bonds of $\partial\Lambda(2\ell)$ where $\sigma^{+,+}$ differs from $\sigma^{-,-}$.
  Taking expectation over the random field finishes the proof.

Step 2: formula for surface tension

- Claim: $\mathbb{E}[\tau_\ell(\eta)] = \frac{2\varepsilon}{\ell}\int_{-\infty}^{\infty} \mathbb{E}[D_\ell(\eta^t)]\,dt$
  with $\eta^t \equiv \eta + \frac{t}{\ell}$ inside $\Lambda(\ell)$,
  $\eta^t \equiv \eta$ outside $\Lambda(\ell)$.
  $D_\ell(\eta^t) := |\{v \in \Lambda(\ell) : \sigma_v^{\Lambda(3\ell),+}(\eta^t) > \sigma_v^{\Lambda(3\ell),-}(\eta^t)\}|$
- Proof: Let $\mathcal{E}^+, \mathcal{E}^-$ be the ground-state energies in $\Lambda(3\ell)$ with +,- boundary conditions, respectively.
- Set $G(\eta) \coloneqq -(\mathcal{E}^+ - \mathcal{E}^-)$
- Then
  $\tau_\ell(\eta) = -(\mathcal{E}^{+,+} + \mathcal{E}^{-,-} - \mathcal{E}^{+,-} - \mathcal{E}^{-,+}) = \lim_{t\to\infty} G(\eta^t) - G(\eta^{-t})$
- Now note that
$$\frac{\partial G}{\partial \eta_v}(\eta) = 2\varepsilon \cdot 1\left\{\sigma_v^{\Lambda(3\ell),+}(\eta) > \sigma_v^{\Lambda(3\ell),-}(\eta)\right\}$$

Step 3: Variance upper bound

- Claim: If $P_\ell \approx \frac{1}{\ell^\delta}$ then $\mathrm{Var}(D_\ell) \le C \cdot \delta \cdot \big(\mathbb{E}(D_\ell)\big)^2$.
- Proof: Write $E_v \coloneqq \left\{\sigma_v^{\Lambda(3\ell),+}(\eta) > \sigma_v^{\Lambda(3\ell),-}(\eta)\right\}$.
- Need to upper bound, for $u, v \in \Lambda(\ell)$,
$$\mathrm{Cov}(1\{E_u\}, 1\{E_v\}) = \mathbb{P}(E_u \cap E_v) - \mathbb{P}(E_u)\mathbb{P}(E_v)$$
- Use $\mathbb{P}(E_u) \ge P_{4\ell} \approx (4\ell)^{-\delta}$
$$\mathbb{P}(E_u \cap E_v) \le \left(P_{\mathrm{dist}(u,v)/2}\right)^2 \approx (\mathrm{dist}(u,v)/2)^{-2\delta}$$
- If δ is small and, say, $\mathrm{dist}(u,v) \ge \ell/100$, get
$$\mathrm{Cov}(1\{E_u\}, 1\{E_v\}) \le \left(\frac{200}{\ell}\right)^{2\delta} - \left(\frac{1}{4\ell}\right)^{2\delta} \approx c \cdot \delta \cdot \ell^{-2\delta}$$
- Can sum such upper bounds to get required result.

Open questions

- Is there a Kosterlitz-Thouless-type transition from exponential to power-law decay of correlations as the random field becomes weaker?
  Mechanism which would imply power-law bound: If the influence percolation behaves like Mandelbrot percolation.
  (connectivity of Mandelbrot percolation – Chayes-Chayes-Durrett)

- For systems with continuous symmetry, such as the random-field XY model, the critical dimension for long-range order is $d_c = 4$ (Imry-Ma 75, Aizenman-Wehr 89).
  Obtain a quantitative decay of correlations there.